The Cameron-Liebler problem for sets

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The outline of the talk

1. Introduction
2. The characterisations result
3. The classification result

- P. J. Cameron and R. A. Liebler, Tactical decompositions and orbits of projective groups, Linear Algebra Appl. 46, 91-102, 1982.
- Cameron and Liebler investigated the orbits of the projective groups $\mathrm{PGL}(n+1, q)$.

Definition

A Cameron-Liebler line class $\mathcal{L}$ with parameter x in $\mathrm{PG}(3, q)$ is a set of $x(q^2+q+1)$ lines in $\mathrm{PG}(3, q)$ such that any line $\ell \in \mathcal{L}$ meets precisely $x(q+1)+q^2-1$ lines of $\mathcal{L}$ in a point and such that any line $\ell \notin \mathcal{L}$ meets precisely $x(q+1)$ lines of $\mathcal{L}$ in a point.

Many equivalent characterisations are known:

A **line spread** of $\mathrm{PG}(3, q)$ is a set of lines that form a partition of the point set of $\mathrm{PG}(3, q)$, i.e. each point of $\mathrm{PG}(3, q)$ is contained in precisely one line of the line spread.

The lines of a line spread are necessarily pairwise skew.

Now a line set $\mathcal{L}$ in $\mathrm{PG}(3, q)$ is a Cameron-Liebler line class with parameter x if and only if it has x lines in common with every line spread of $\mathrm{PG}(3, q)$.

The central problem for Cameron-Liebler line classes in $\mathrm{PG}(3, q)$ is to determine for which parameters x a Cameron-Liebler line class exists, and to classify the examples admitting a given parameter x.

$\mathrm{PG}(3, q)$: a complete classification is not finished

$\mathrm{PG}(2k+1, q)$: recently, Cameron-Liebler k-classes in $\mathrm{PG}(2k+1, q)$ were introduced by M. Rodgers, L. Storme and A. Vansweevelt, and Cameron-Liebler line classes in $\mathrm{PG}(n, q)$ were studied by A. L. Gavrilyuk and I. Y. Mogilnykh.

A subset of size k of a set will be called shortly a *k-subset*.

Definition

A k-**uniform partition** of a finite set Ω, with $|\Omega| = n$ and $k \mid n$, is a set of pairwise disjoint k-subsets of Ω such that any element of Ω is contained in precisely one of the k-subsets.

Necessarily, a k-uniform partition of a finite set Ω, with $|\Omega| = n$, contains $\frac{n}{k}$ different k-subsets.

Definition

Let Ω be a finite set with $|\Omega| = n$ and let k be a divisor of n. A **Cameron-Liebler class of k-sets with parameter** x is a set of k-subsets of Ω which has x different k-subsets in common with every k-uniform partition of Ω.

The next result is the Erdős-Ko-Rado theorem, a classical result in combinatorics.

Theorem

If $\mathcal{S}$ is a family of k-subsets in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of $\mathcal{S}$ are pairwise not disjoint, then $|\Omega| \leq \binom{n-1}{k-1}$. Moreover, if $n \geq 2k+1$, then equality holds if and only if $\mathcal{S}$ is the set of all k-subsets through a fixed element of Ω.

Lemma

Let Ω be a finite set with $|\Omega| = n$, and let $\mathcal{L}$ be a Cameron-Liebler class of k-sets with parameter x in Ω, with $k \mid n$.

1. *The number of k-uniform partitions of Ω equals* $\dfrac{n!}{\left(\frac{n}{k}\right)!(k!)^{\frac{n}{k}}}$.
2. *The number of k-sets in $\mathcal{L}$ equals* $x\binom{n-1}{k-1}$.
3. *The set $\overline{\mathcal{L}}$ of k-subsets of Ω not belonging to $\mathcal{L}$ is a Cameron-Liebler class of k-sets with parameter* $\frac{n}{k} - x$.

Example

Let Ω be a finite set with $|\Omega| = n$, and assume $k \mid n$. We give some examples of Cameron-Liebler classes of k-sets with parameter x. Note that $0 \leq x \leq \frac{n}{k}$.

- The empty set is obviously a Cameron-Liebler class of k-sets with parameter 0.
- The set of all k-subsets of Ω is a Cameron-Liebler class of k-sets with parameter $\frac{n}{k}$.
- These two examples are called the **trivial** Cameron-Liebler classes of k-sets.

Example

- Let p be a given element of Ω. The set of k-subsets of Ω containing p is a Cameron-Liebler class of k-sets with parameter 1.
- The set of all k-subsets of Ω not containing the element p is a Cameron-Liebler class of k-sets with parameter $\frac{n}{k} - 1$.

The **incidence vector** of a subset A of a set S is the vector whose positions correspond to the elements of S, with a one on the positions corresponding to an element in A and a zero on the other positions.

Below we will use the incidence vector of a family of k-subsets of a set Ω: as this family is a subset of the set of all k-subsets of Ω, each position corresponds to a k-subset of Ω.

For any vector v whose positions correspond to elements in a set, we denote its value on the position corresponding to an element a by $(v)_a$. The all-one vector will be denoted by j.

Given a set Ω, we also define the **incidence matrix of elements and k-subsets**.

This is the $|\Omega| \times \binom{|\Omega|}{k}$-matrix whose rows are labelled with the elements of Ω and whose columns are labelled with the k-sets of Ω and whose entries equal 1 if the element corresponding to the row is contained in the k-set corresponding to the column, and zero otherwise.

The **Kneser matrix** or **disjointness matrix** of k-sets in Ω is the $\binom{|\Omega|}{k} \times \binom{|\Omega|}{k}$-matrix whose rows and columns are labelled with the k-sets of Ω and whose entries equal 1 if the k-set corresponding to the row and the k-set corresponding to the column are disjoint, and zero otherwise.

Lemma

Let Ω be a finite set with $|\Omega| = n$ and let K be the Kneser matrix of the k-sets in Ω. The eigenvalues of K are given by $\lambda_j = (-1)^j \binom{n-k-j}{k-j}$, $j = 0, \ldots, k$, and the multiplicity of the eigenvalue λ_j is $\binom{n}{j} - \binom{n}{j-1}$.

Now we can present a theorem with many equivalent characterisations of Cameron-Liebler classes of k-subsets.

Theorem

Let Ω be a finite set with $|\Omega| = n$, and let k be a divisor of n. Let $\mathcal{L}$ be a set of k-subsets of Ω with incidence vector χ. Denote $\frac{|\mathcal{L}|}{\binom{n-1}{k-1}}$ by x. Let C be the incidence matrix of elements and k-subsets in Ω and let K be the Kneser matrix of k-sets in Ω. The following statements are equivalent.

- (i) *$\mathcal{L}$ is a Cameron-Liebler class of k-sets with parameter x.*
- (ii) *$\mathcal{L}$ has x different k-subsets in common with every k-uniform partition of Ω.*
- (iii) *For each fixed k-subset π of Ω, the number of elements of $\mathcal{L}$ disjoint from π equals $(x - (\chi)_\pi)\binom{n-k-1}{k-1}$.*
- (iv) *The vector $\chi - \frac{kx}{n} j$ is contained in the eigenspace of K for the eigenvalue $-\binom{n-k-1}{k-1}$.*
- (v) $\chi \in \mathrm{row}(C)$.
- (vi) $\chi \in (\ker(C))^\perp$.

Theorem

Let Ω be a finite set with $|\Omega| = n$ and let $\mathcal{L}$ be a Cameron-Liebler class of k-sets with parameter x in Ω, $k \geq 2$. If $n \geq 3k$ and $\mathcal{L}$ is nontrivial, then either $x = 1$ and $\mathcal{L}$ is the set of all k-subsets containing a fixed element or $x = \frac{n}{k} - 1$ and $\mathcal{L}$ is the set of all k-subsets not containing a fixed element.

The theorem states that the examples shown before are the only examples of Cameron-Liebler classes of k-sets, in case $n \geq 3k$.

Only four parameter values are admissable.

Lemma

Let $\mathcal{L}$ be a nontrivial Cameron-Liebler class of k-sets with parameter x in a set Ω of size $n \geq 3k$, $x < \frac{n}{k} - 1$ and $k \geq 2$. Then, $\mathcal{L}$ is the set of all k-sets through a fixed element and $x = 1$.

Lemma

Let $\mathcal{L}$ be a Cameron-Liebler class of k-sets with parameter $\frac{n}{k} - 1$ in a set Ω of size $n \geq 3k$, with $k \geq 2$. Then, $\mathcal{L}$ is the set of all k-sets not through a fixed element.

Remark

Let Ω be a set of size n, and let k be a divisor of n. The main Theorem does not cover the cases $k = 1$, and $n \in \{k, 2k\}$.

- *Assume $k = 1$, then any set of x different 1-subsets of Ω is a Cameron-Liebler class of k-sets with parameter x. So, in this case each value x, with $0 \leq x \leq n$, is admissable as parameter of a Cameron-Liebler class.*
- *If $n = k$, there is only one subset of size k, and thus all Cameron-Liebler classes of k-sets are trivial.*

Remark

- *If $n = 2k$, each k-uniform partition consists of two k-sets which are the complement of each other.*
- *Every set of k-subsets that is constructed by picking one of both k-sets from each k-uniform partition, is a Cameron-Liebler class of k-sets with parameter 1.*

Thank you for your attention!