On mixed dimension subspace codes

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Motivation

Let V be an n–dimensional vector space over $\mathrm{GF}(q)$, q prime power,

$(\mathcal{S}(V), d_s)$, $(\mathcal{G}_q(n,k), d_s)$ are metric spaces

- $\mathcal{S}(V)$ set of all subspaces of $\mathrm{PG}(n-1,q)$,
- $\mathcal{G}_q(n,k)$ set of all k–dimensional subspaces of $\mathrm{PG}(n-1,q)$, *Grassmannian*,
- $d_s(U,U') = \dim(U+U') - \dim(U \cap U')$ *subspace distance*.

The main problem in subspace coding theory

- determination of the maximum size of codes with given minimum distance,
- the classification of the corresponding optimal codes.

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- T. Etzion, Problems on q-analogs in coding theory, *preprint* (arXiv:1305.6126).

q-analogs

- subspace codes are q-analogs of constant weight codes
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$\mathcal{A}_q(n, d)$ the maximum size of an $(n, M, d)_q$ mixed–dimension subspace code.

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$q^3 + 1$

largest partial line spread of $\mathrm{PG}(4, q)$

$\mathcal{C}$ optimal $(5, 3)_q$ subspace code

lines contained in $\mathcal{C}$ are pairwise disjoint

$\mathcal{C}$ contains at most $q^3 + 1$ lines

Dual argument

planes contained in $\mathcal{C}$ are pairwise intersecting in a point

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if $\mathcal{C}$ consists of lines and planes

$$|\mathcal{C}| \leq 2(q^3 + 1).$$

Properties

$\mathcal{C}$ contains at most one point,

$\mathcal{C}$ contains at most one solid,

if $\mathcal{C}$ contains one point, then $\mathcal{C}$ contains at most q^3 planes,

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$$\mathcal{A}_q(5,3) \leq 2(q^3 + 1)$$

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The odd characteristic case

$q = p^h$ odd prime power

$\mathrm{PG}(4, q)$

$$\ell : X_3 = X_4 = X_5 = 0,$$
$$\pi : X_4 = X_5 = 0,$$
$$\Sigma_i : X_4 = \omega^{i-1} X_5,\ 1 \le i \le q-1,$$
$$\Sigma_q : X_4 = 0,$$
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$$\ell \subset \pi \subset \Sigma_i$$

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The odd characteristic case

$$a, b, c \in \mathrm{GF}(q)$$

$X^3 + aX^2 + bX + c = 0$ is irreducible over $\mathrm{GF}(q)$,

$$M_{r,s,t} = \begin{pmatrix} 1 & 0 & r & r^2 - ar + s & t \\ 0 & 1 & s & 2rs - t & s^2 + bs - cr \\ 0 & 0 & 1 & 2r & 2s \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \{M_{r,s,t} \mid r, s, t \in \mathrm{GF}(q)\} \leq \mathrm{PGL}(5, q),$$

p–group of order q^3.

Action of G on points of $\mathrm{PG}(4, q)$

- points of ℓ,
- $\pi \setminus \ell$,
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$$X^3 + aX^2 + bX + c = 0 \text{ is irreducible over } \mathrm{GF}(q),$$

$$M_{r,s,t} = \begin{pmatrix} 1 & 0 & r & r^2 - ar + s & t \\ 0 & 1 & s & 2rs - t & s^2 + bs - cr \\ 0 & 0 & 1 & 2r & 2s \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$G = \{M_{r,s,t} \mid r, s, t \in \mathrm{GF}(q)\} \leq \mathrm{PGL}(5, q),$$

p–group of order q^3.

Action of G on points of $\mathrm{PG}(4, q)$

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a) ℓ,

b) $q+1$ orbits of size q, lines of π through a point of ℓ,

c) $(q+1)^2$ orbits of size q^2, lines of $\Sigma_i \setminus \pi$ through a point of ℓ, $1 \leq i \leq q+1$,

d) $q(q+1)$ orbits of size q^3, lines of Σ_i disjoint from ℓ, $1 \leq i \leq q+1$,

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A line–orbit of type e) is a partial line spread of $\mathrm{PG}(4, q)$.

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σ plane,

$\sigma \cap \pi \notin \ell$, $\sigma \notin \Sigma_i$

σ^G set of q^3 planes mutually intersecting in a point

$\mathcal{L}$ line–orbit of type e),

no line of $\mathcal{L}$ is contained in σ

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$\ell : X_1 = X_4 = X_5 = 0,$

$\pi : X_4 = X_5 = 0,$

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pencil $\mathcal{F}$

- solid Σ,
- cone $\mathcal{C}$: vertex $N = (1, 0, 0, 0, 0)$

  base $\mathcal{H}$ hyperbolic quadric of Σ

  $X_2X_5 + X_3X_4 = 0$
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$$G = \{M_{a,b,c,d} \mid a, b, c, d \in \mathrm{GF}(q), a \neq 0, c^2 + cd + \alpha d^2 = 1\}$$

$$G \simeq C_{q+1} \times (E_q \times C_{q-1}) \leq \mathrm{PGL}(5, q)$$
$$|G| = q^3 - q$$
G fixes each quadric of $\mathcal{F}$

$\Sigma \setminus \mathcal{H}$, $\mathcal{C} \setminus (\pi \cup \mathcal{H})$, $\mathcal{Q}_i \setminus \mathcal{H}$, $1 \leq i \leq q-1$

r line meeting each of $\Sigma \setminus \mathcal{H}$, $\mathcal{C} \setminus (\pi \cup \mathcal{H})$, $\mathcal{Q}_i \setminus \mathcal{H}$, $1 \leq i \leq q-1$ in exactly one point,

r^G set of $q^3 - q$ lines forming partial spread

there are $q^2 - q$ line–orbits of this type

there are $q^2 - q$ plane–orbits of size $q^3 - q$ consisting of planes mutually intersecting in one point.

The even characteristic case

$\Sigma \setminus \mathcal{H}$, $\mathcal{C} \setminus (\pi \cup \mathcal{H})$, $\mathcal{Q}_i \setminus \mathcal{H}$, $1 \leq i \leq q-1$

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The even characteristic case

$\mathcal{R}_1$ regulus of $\mathcal{H}$ containing ℓ, $\mathcal{R}_2$ opposite regulus,

$\mathcal{P}_1$ be a plane–orbit of size $q^3 - q$

there exists a line–orbit $\mathcal{L}$ of size $q^3 - q$

no line of $\mathcal{L}$ is contained in a plane of $\mathcal{P}_1$

$\mathcal{P}_2$ set of $q + 1$ planes generated by a line of $\mathcal{R}_1$ and the point N

$\mathcal{P}_1 \cup \mathcal{P}_2$ set of $q^3 + 1$ planes mutually intersecting in a point

$\mathcal{L} \cup \mathcal{R}_2$ set of size $q^3 + 1$ disjoint lines

$\mathcal{L} \cup \mathcal{R}_2 \cup \mathcal{P}_1 \cup \mathcal{P}_2$

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$\mathcal{P}_1 \cup \mathcal{P}_2$ set of q^3+1 planes mutually intersecting in a point

$\mathcal{L} \cup \mathcal{R}_2$ set of size q^3+1 disjoint lines

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- T. Etzion, A. Vardy, Error-correcting codes in projective space, *IEEE Trans. Inform. Theory* 57 (2011), no. 2, 1165-1173.

$$\mathcal{A}_q(5,3) = 2(q^3+1)$$

- T. Honold, M. Kiermaier, S. Kurz, Constructions and Bounds for Mixed-Dimension Subspace Codes, *preprint arXiv:1512.06660.*

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THANK YOU