

# New quasi-symmetric designs by the Kramer-Mesner method<sup>\*</sup>

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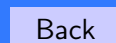
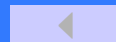
*and*

Renata Vlahović

University of Zagreb, Croatia

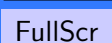
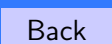
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<sup>\*</sup> Supported in part by the Croatian Science Foundation under project 1637 and by the European Science Foundation under COST Action IC1104.





A  $t$ - $(v, k, \lambda)$  **design** is a set  $V = \{1, \dots, v\}$  of **points** together with a family  $\mathcal{B} = \{B_1, \dots, B_b\}$  of  $k$ -element subsets of  $V$  called **blocks** such that any  $t$ -subset of points is contained in exactly  $\lambda$  blocks.





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The total number of blocks  $b$  and the number of blocks  $r$  through any given point can be computed from  $t$ ,  $v$ ,  $k$  and  $\lambda$ .

$$b = \lambda \cdot \frac{\binom{v}{t}}{\binom{k}{t}}, \quad r = \lambda \cdot \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$$



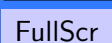
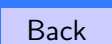


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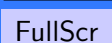
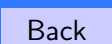
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An **automorphism** of the design is a permutation  $\alpha : V \rightarrow V$  taking blocks to blocks, i.e. such that  $\alpha(B) \in \mathcal{B}$  for any  $B \in \mathcal{B}$ .



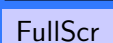
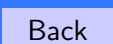
Let  $G$  be a group of permutations of  $V$  and let  $\mathcal{T}_1, \dots, \mathcal{T}_m$  be the orbits of  $t$ -element subsets of  $V$  and  $\mathcal{K}_1, \dots, \mathcal{K}_n$  the orbits of  $k$ -element subsets of  $V$ .





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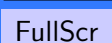
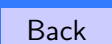




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The matrix  $A = [a_{ij}]$  is the **Kramer-Mesner matrix**. Designs with  $G$  as an automorphism group correspond to 0–1 solutions of the system of linear equations  $A \cdot x = \lambda J$ , where  $J = (1, \dots, 1)^T$ .



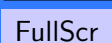
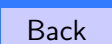


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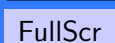
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Solving systems of linear equations over the integers is a NP complete problem. The Kramer-Mesner system is computationally feasible only if the number of variables  $n$  is sufficiently small.





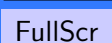
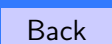
A  $t$ -( $v, k, \lambda$ ) design is **quasi-symmetric** if any two blocks intersect either in  $x$  or in  $y$  points, for non-negative integers  $x < y$ . Quasi-symmetric designs have important connections with strongly regular graphs and self-orthogonal codes.



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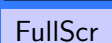
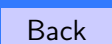
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- for  $t = 4$  the only example is the  $4$ - $(23, 7, 1)$  design ( $x = 1, y = 3$ ) and its complement;

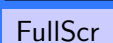
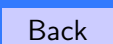




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- for  $t = 3$  it is conjectured that the only examples are
  - the quasi-symmetric 4-design and its residual 3- $(22, 7, 4)$  design,
  - Hadamard 3-designs,
  - 3- $((\lambda + 1)(\lambda^2 + 5\lambda + 5), (\lambda + 1)(\lambda + 2), \lambda)$  designs (known to exist only for  $\lambda = 1$ ),
  - a hypothetical 3- $(496, 40, 3)$  design,
  - complements of these designs.



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No.	$v$	$k$	$\lambda$	$r$	$b$	$x$	$y$	Existence
1	19	7	7	21	57	1	3	No
2	19	9	16	36	76	3	5	No
3	20	10	18	38	76	4	6	No
4	20	8	14	38	95	2	4	No
5	21	9	12	30	70	3	5	No
6	21	8	14	40	105	2	4	No
7	21	6	4	16	56	0	2	Yes(1)
8	21	7	12	40	120	1	3	Yes(1)
9	22	8	12	36	99	2	4	No
10	22	6	5	21	77	0	2	Yes(1)
11	22	7	16	56	176	1	3	Yes(1)
12	23	7	21	77	253	1	3	Yes(1)
13	24	8	7	23	69	2	4	No
14	28	7	16	72	288	1	3	No
15	28	12	11	27	63	4	6	Yes( $\geq 8784$ )
16	29	7	12	56	232	1	3	No
17	31	7	7	35	155	1	3	Yes(5)
18	33	15	35	80	176	6	9	?
19	33	9	6	24	88	1	3	No
20	35	7	3	17	85	1	3	No
21	35	14	13	34	85	5	8	?
22	36	16	12	28	63	6	8	Yes( $\geq 8784$ )
23	37	9	8	36	148	1	3	?
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25	41	9	9	45	205	1	3	?
26	41	20	57	120	246	8	11	?
27	41	17	34	85	205	5	8	No
28	42	21	60	123	246	9	12	?
29	42	18	51	123	287	6	9	?
30	43	18	51	126	301	6	9	No
31	43	16	40	112	301	4	7	?
32	45	21	70	154	330	9	13	?
33	45	9	8	44	220	1	3	Yes(1)
34	45	18	34	88	220	6	9	?
35	45	15	42	132	396	3	6	?
36	46	16	72	216	621	4	7	?
37	46	16	8	24	69	4	6	?
38	49	9	6	36	196	1	3	Yes( $\geq 44$ )
39	49	16	45	144	441	4	7	?
40	49	13	13	52	196	1	4	?
41	51	21	14	35	85	6	9	No
42	51	15	7	25	85	3	5	No
43	52	16	20	68	221	4	7	?
44	55	16	40	144	495	4	8	?
45	55	15	63	243	891	3	6	?
46	55	15	7	27	99	3	5	?
47	56	16	18	66	231	4	8	?
48	56	15	42	165	616	3	6	?



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## Adaptations of Kramer-Mesner method to quasi-symmetric designs:

1. An orbit of  $k$ -subsets  $\mathcal{K}_i$  is **good** if  $|K_1 \cap K_2|$  is either  $x$  or  $y$ , for any two elements  $K_1, K_2 \in \mathcal{K}_i$ . We can limit the search to good orbits and thereby reduce the number of variables  $n$ .



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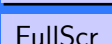
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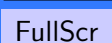
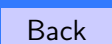
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2. Two orbits  $\mathcal{K}_i, \mathcal{K}_j$  are **compatible** if  $|K_1 \cap K_2|$  is either  $x$  or  $y$ , for any  $K_1 \in \mathcal{K}_i, K_2 \in \mathcal{K}_j$ . The  $n \times n$  matrix  $C = [c_{ij}]$  with  $c_{ij} = 1$  if  $\mathcal{K}_i$  and  $\mathcal{K}_j$  are compatible, and  $c_{ij} = 0$  otherwise, is the **compatibility matrix**. This information can be used to make the backtracking search for solutions of the Kramer-Mesner system more efficient.



**Example:**  $2-(28, 12, 11)$ ,  $x = 4$ ,  $y = 6$ . Group  $G = \langle \alpha, \beta \rangle$  isomorphic to the dihedral group of order 12 generated by the permutations

$$\alpha = (1, 2, 3, 4, 5, 6)(7, 8, 9, 10, 11, 12)(13, 14, 15, 16, 17, 18)(19, 20, 21, 22, 23, 24)(25, 26, 27),$$

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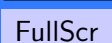
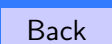
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$$V = \{1, \dots, 28\}$$

Number of orbits  $\mathcal{T}_1, \dots, \mathcal{T}_m$  of 2-element subsets:  $m = 47$ .



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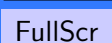
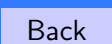
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Total number of orbits of 12-element subsets:  $n = 2\,543\,568$ .



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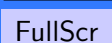
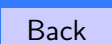
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Number of orbits  $\mathcal{T}_1, \dots, \mathcal{T}_m$  of 2-element subsets:  $m = 47$ .

Total number of orbits of 12-element subsets:  $n = 2\,543\,568$ .

Number of good orbits  $\mathcal{K}_1, \dots, \mathcal{K}_n$ :  $n = 1097$ .



**Example:**  $2-(28, 12, 11)$ ,  $x = 4$ ,  $y = 6$ . Group  $G = \langle \alpha, \beta \rangle$  isomorphic to the dihedral group of order 12 generated by the permutations

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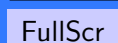
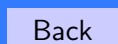
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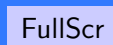
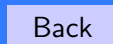
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We use **GAP** to compute the orbits and set up the Kramer-Mesner system, our own backtracking **solver** written in C, and **nauty2** by B. D. McKay and A. Piperno for isomorphism checking.





**Proposition.** Up to isomorphism there are 13656 quasi-symmetric  $(28, 12, 11)$  designs with  $x = 4$ ,  $y = 6$  and  $G = \langle \alpha, \beta \rangle$  as an automorphism group.

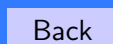


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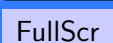
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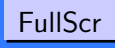
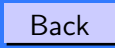
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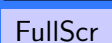
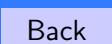
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**Theorem.** There are more than 50 000 quasi-symmetric  $(28, 12, 11)$  designs and more than 500 000 quasi-symmetric  $(36, 16, 12)$  designs up to isomorphism.



**Example:**  $2-(56, 16, 18)$ ,  $x = 4$ ,  $y = 8$  (unknown).

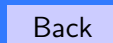
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The associated strongly regular graph has parameters  $SRG(231, 30, 9, 3)$ . Such a graph exists (the *Cameron graph*) and has a group of automorphisms isomorphic to the Mathieu group  $M_{21}$  of order 20160.



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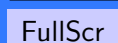
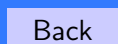
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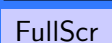
The group  $M_{21}$  also acts on the quasi-symmetric  $2-(21, 6, 4)$ ,  $x = 0$ ,  $y = 2$  design with  $b = 56$  blocks. It has a subgroup  $G$  of order 960 isomorphic to  $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cdot A_5$ . We take the permutation representation of degree 56 determined by the action on the blocks of the  $(21, 6, 4)$  design:  $G = \langle \alpha, \beta \rangle$ ,

$$\alpha = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)(16, 17, 18, 19, 20)(21, 22, 23, 24, 25)(26, 27, 28, 29, 30) \\ (31, 32, 33, 34, 35)(36, 37, 38, 39, 40)(41, 42, 43, 44, 45)(46, 47, 48, 49, 50)(51, 52, 53, 54, 55),$$

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**Theorem.** There are three quasi-symmetric  $(56, 16, 18)$  designs with  $x = 4$ ,  $y = 8$  and  $G = \langle \alpha, \beta \rangle$  as an automorphism group. The first one has  $G$  as its full automorphism group, the second one has a split extension  $G.\mathbb{Z}_2$  of order 1920, and the third one has a split extension  $M_{21}.\mathbb{Z}_2$  of order 40320 as its full automorphism group.

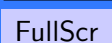
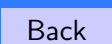




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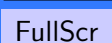
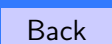




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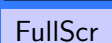
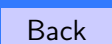




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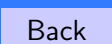




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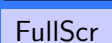
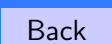




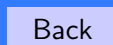
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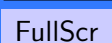
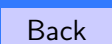


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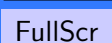
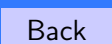
Description from M. Grassl's page [www.codetables.de](http://www.codetables.de) based on A. E. Brouwer's tables:

Construction of a linear code  $[56, 19, 16]$  over  $GF(2)$ :

- 1:  $[55, 21, 15]$  Cyclic Linear Code over  $GF(2)$ . CyclicCode of length 55 with generating polynomial  $x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{23} + x^{19} + x^{18} + x^{13} + x^{10} + x^7 + x^5 + x^3 + x + 1$ .
- 2:  $[56, 21, 16]$  Linear Code over  $GF(2)$ . ExtendCode [1] by 1.
- 3:  $[56, 19, 16]$  Linear Code over  $GF(2)$ . Subcode of [2].



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Back

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**Thanks for your attention!**

