New quasi-symmetric designs by the Kramer-Mesner method*

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A t-(v,k,λ) design is a set $V = \{1,\ldots,v\}$ of points together with a family $\mathcal{B} = \{B_1,\ldots,B_b\}$ of k-element subsets of V called blocks such that any t-subset of points is contained in exactly λ blocks.

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The total number of blocks b and the number of blocks r through any given point can be computed from t, v, k and λ.

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An automorphism of the design is a permutation $\alpha : V\longrightarrow V$ taking blocks to blocks, i.e. such that $\alpha(B)\in\mathcal{B}$ for any $B\in\mathcal{B}$.

Let G be a group of permutations of V and let $\mathcal{T}_1, \ldots, \mathcal{T}_m$ be the orbits of t-element subsets of V and $\mathcal{K}_1, \ldots, \mathcal{K}_n$ the orbits of k-element subsets of V.

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The matrix $A = [a_{ij}]$ is the Kramer-Mesner matrix. Designs with G as an automorphism group correspond to 0–1 solutions of the system of linear equations $A \cdot x = \lambda\, J$, where $J = (1, \ldots, 1)^\tau$.

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Solving systems of linear equations over the integers is a NP complete problem. The Kramer-Mesner system is computationally feasible only if the number of variables n is sufficiently small.

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- do not exist for $t \geq 5$;
- for $t = 4$ the only example is the 4-$(23,7,1)$ design ($x = 1$, $y = 3$) and its complement;
- for $t = 3$ it is conjectured that the only examples are
  - the quasi-symmetric 4-design and its residual 3-$(22,7,4)$ design,
  - Hadamard 3-designs,
  - 3-$((\lambda+1)(\lambda^2+5\lambda+5),(\lambda+1)(\lambda+2),\lambda)$ designs (known to exist only for $\lambda = 1$),
  - a hypothetical 3-$(496,40,3)$ design,
  - complements of these designs.

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No.	v	k	λ	r	b	x	y	Existence
1	19	7	7	21	57	1	3	No
2	19	9	16	36	76	3	5	No
3	20	10	18	38	76	4	6	No
4	20	8	14	38	95	2	4	No
5	21	9	12	30	70	3	5	No
6	21	8	14	40	105	2	4	No
7	21	6	4	16	56	0	2	Yes(1)
8	21	7	12	40	120	1	3	Yes(1)
9	22	8	12	36	99	2	4	No
10	22	6	5	21	77	0	2	Yes(1)
11	22	7	16	56	176	1	3	Yes(1)
12	23	7	21	77	253	1	3	Yes(1)
13	24	8	7	23	69	2	4	No
14	28	7	16	72	288	1	3	No
15	28	12	11	27	63	4	6	Yes(≥ 8784)
16	29	7	12	56	232	1	3	No
17	31	7	7	35	155	1	3	Yes(5)
18	33	15	35	80	176	6	9	?
19	33	9	6	24	88	1	3	No
20	35	7	3	17	85	1	3	No
21	35	14	13	34	85	5	8	?
22	36	16	12	28	63	6	8	Yes(≥ 8784)
23	37	9	8	36	148	1	3	?
24	39	12	22	76	247	3	6	?

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25	41	9	9	45	205	1	3	?
26	41	20	57	120	246	8	11	?
27	41	17	34	85	205	5	8	No
28	42	21	60	123	246	9	12	?
29	42	18	51	123	287	6	9	?
30	43	18	51	126	301	6	9	No
31	43	16	40	112	301	4	7	?
32	45	21	70	154	330	9	13	?
33	45	9	8	44	220	1	3	Yes(1)
34	45	18	34	88	220	6	9	?
35	45	15	42	132	396	3	6	?
36	46	16	72	216	621	4	7	?
37	46	16	8	24	69	4	6	?
38	49	9	6	36	196	1	3	Yes(≥ 44)
39	49	16	45	144	441	4	7	?
40	49	13	13	52	196	1	4	?
41	51	21	14	35	85	6	9	No
42	51	15	7	25	85	3	5	No
43	52	16	20	68	221	4	7	?
44	55	16	40	144	495	4	8	?
45	55	15	63	243	891	3	6	?
46	55	15	7	27	99	3	5	?
47	56	16	18	66	231	4	8	?
48	56	15	42	165	616	3	6	?

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Adaptations of Kramer-Mesner method to quasi-symmetric designs:

1. An orbit of k-subsets $\mathcal{K}_i$ is good if $|K_1 \cap K_2|$ is either x or y, for any two elements $K_1, K_2 \in \mathcal{K}_i$. We can limit the search to good orbits and thereby reduce the number of variables n.

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2. Two orbits $\mathcal{K}_i$, $\mathcal{K}_j$ are compatible if $|K_1 \cap K_2|$ is either x or y, for any $K_1 \in \mathcal{K}_i$, $K_2 \in \mathcal{K}_j$. The $n \times n$ matrix $C = [c_{ij}]$ with $c_{ij} = 1$ if $\mathcal{K}_i$ and $\mathcal{K}_j$ are compatible, and $c_{ij} = 0$ otherwise, is the compatibility matrix. This information can be used to make the backtracking search for solutions of the Kramer-Mesner system more efficient.

Example: 2-$(28, 12, 11)$, $x = 4$, $y = 6$. Group $G = \langle \alpha, \beta \rangle$ isomorphic to the dihedral group of order 12 generated by the permutations

$$\alpha = (1,2,3,4,5,6)(7,8,9,10,11,12)(13,14,15,16,17,18)(19,20,21,22,23,24)(25,26,27),$$

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We use **GAP** to compute the orbits and set up the Kramer-Mesner system, our own backtracking **solver** written in C, and **nauty2** by B. D. McKay and A. Piperno for isomorphism checking.

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Theorem. There are more than $50\,000$ quasi-symmetric $(28, 12, 11)$ designs and more than $500\,000$ quasi-symmetric $(36, 16, 12)$ designs up to isomorphism.

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How to choose the group G?

The associated strongly regular graph has parameters $SRG(231,30,9,3)$. Such a graph exists (the *Cameron graph*) and has a group of automorphisms isomorphic to the Mathieu group M_{21} of order 20160.

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The associated strongly regular graph has parameters $SRG(231, 30, 9, 3)$. Such a graph exists (the *Cameron graph*) and has a group of automorphisms isomorphic to the Mathieu group M_{21} of order 20160.

The group M_{21} also acts on the quasi-symmetric 2-$(21, 6, 4)$, $x = 0$, $y = 2$ design with $b = 56$ blocks. It has a subgroup G of order 960 isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2).A_5$. We take the permutation representation of degree 56 determined by the action on the blocks of the $(21, 6, 4)$ design: $G = \langle \alpha, \beta \rangle$,

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Theorem. There are three quasi-symmetric $(56, 16, 18)$ designs with $x = 4$, $y = 8$ and $G = \langle \alpha, \beta \rangle$ as an automorphism group. The first one has G as its full automorphism group, the second one has a split extension $G.\mathbb{Z}_2$ of order 1920, and the third one has a split extension $M_{21}.\mathbb{Z}_2$ of order 40320 as its full automorphism group.

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- there are 1242 "short orbits", of size $\leq b$; we can generate them directly by an algorithm written in GAP

Theorem. There are three quasi-symmetric $(56, 16, 18)$ designs with $x = 4$, $y = 8$ and $G = \langle \alpha, \beta \rangle$ as an automorphism group. The first one has G as its full automorphism group, the second one has a split extension $G.\mathbb{Z}_2$ of order 1920, and the third one has a split extension $M_{21}.\mathbb{Z}_2$ of order 40320 as its full automorphism group.

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- the 7×40 Kramer-Mesner system has 5 solutions respecting the compatibility matrix, giving rise to the 3 quasi-symmetric designs described in the theorem

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Description from M. Grassl's page `www.codetables.de` based on A. E. Brouwer's tables:

Construction of a linear code $[56, 19, 16]$ over $GF(2)$:

1: $[55, 21, 15]$ Cyclic Linear Code over $GF(2)$. CyclicCode of length 55 with generating polynomial $x^{34} + x^{31} + x^{29} + x^{28} + x^{26} + x^{23} + x^{19} + x^{18} + x^{13} + x^{10} + x^7 + x^5 + x^3 + x + 1$.

2: $[56, 21, 16]$ Linear Code over $GF(2)$. ExtendCode [1] by 1.

3: $[56, 19, 16]$ Linear Code over $GF(2)$. Subcode of [2].

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Idea: look at minimum weight words in the $[56, 21, 16]_2$ code from step 2. There are 5170 words of weigt 16; quasi-symmetric designs are equivalent to $b = 231$ words pairwise intersecting in $x = 4$ or $y = 8$ coordinates.

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Thanks for your attention!